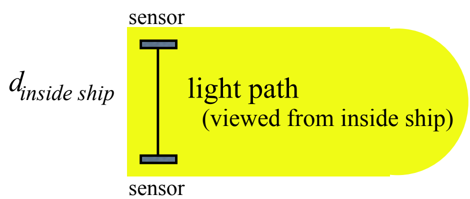
TIME DILATION and LENGTH CONTRACTION

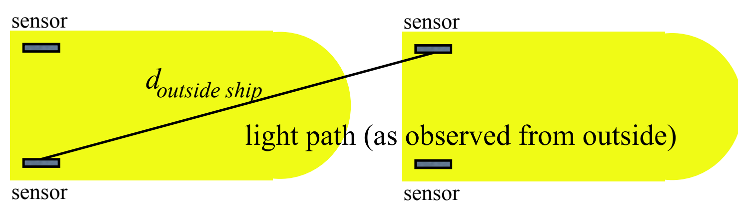
from “inside” frame of reference

$\Delta t_{inside\ ship}$ is time of transit measured by a clock inside the ship as viewed by someone “looking in” from outside the ship

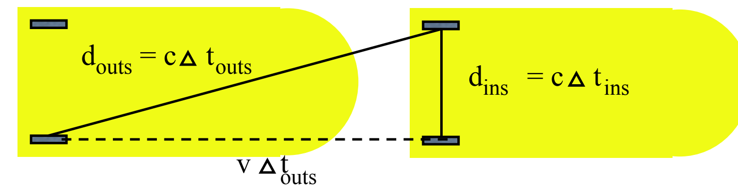


from “outside” frame of reference

$\Delta t_{outside\ ship}$ is time of transit measured by a clock and observer outside the ship



looking at the right triangle



Using the Pythagorean relationship:

TIME DILATION

$$(d_{outside})^2 = (d_{inside})^2 + (v_{rel}\Delta t_{outside})^2$$

$$\Rightarrow (c\Delta t_{outside})^2 = (c\Delta t_{inside})^2 + (v_{rel}\Delta t_{outside})^2$$

$$\Rightarrow (\Delta t_{outside})^2 - \left(\frac{v_{rel}}{c}\Delta t_{outside}\right)^2 = (\Delta t_{inside})^2$$

$$\Rightarrow (\Delta t_{outside})^2\left(1-\left(\frac{v_{rel}}{c}\right)^2\right) = (\Delta t_{inside})^2$$

$$\Rightarrow (\Delta t_{outside})\left(1-\left(\frac{v_{rel}}{c}\right)^2\right)^{1/2} = (\Delta t_{inside})$$

$$\Rightarrow (\Delta t_{outside}) = (\Delta t_{inside})\frac{1}{\sqrt{1-\left(\frac{v_{rel}}{c}\right)^2}}$$

$$\Rightarrow (\Delta t_{outside}) = (\Delta t_{inside})\gamma$$

RELATIVISTIC FACTOR (LORENTZ FACTOR)

There are a couple of things to notice about this relationship.

1.) The "Lorentz factor," sometimes called "the relativistic factor," is defined as:

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v_{rel}}{c}\right)^2}}$$

This quantity defines the amount of deviation that exists between the two clocks.

NOTE that the relativistic factor is always GREATER THAN ONE.

2.) In some cases, the relativistic factor is written as:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

where the beta term is the relative velocity in terms of the speed of light. That is,

$$\beta = \frac{v_{rel}}{c} = \text{a dimensionless number less than 1}$$

TIME DILATION

$$\left(\Delta t_{\substack{\text{measured by clock} \\ \text{in outside frame as} \\ \text{observed from outside}}}\right) = \left(\Delta t_{\substack{\text{measured by clock} \\ \text{in inside frame as} \\ \text{observed from outside} \\ \text{(and inside)}}}\right)\frac{1}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \left(\Delta t_{\substack{\text{measured by clock} \\ \text{in outside frame as} \\ \text{observed from outside}}}\right) = \gamma\left(\Delta t_{\substack{\text{measured by clock} \\ \text{in inside frame as} \\ \text{observed from outside}}}\right)$$

LENGTH CONTRACTION

$$\left(L_{\substack{\text{measured by meter stick} \\ \text{in outside frame as} \\ \text{observed from outside}}}\right) = \left(L_{\substack{\text{measured by meter stick} \\ \text{in inside frame as} \\ \text{observed from outside}}}\right)\sqrt{1-\beta^2}$$

$$\Rightarrow \left(L_{\substack{\text{measured by meter stick} \\ \text{in outside frame as} \\ \text{observed from outside}}}\right) = \frac{1}{\gamma}\left(L_{\substack{\text{measured by meter stick} \\ \text{in inside frame as} \\ \text{observed from both} \\ \text{outside and inside}}}\right)$$

RELATIVE SPEEDS

A space ship moving at v passes a second ship moving in the opposite direction with velocity u. What relative speed w does the first ship register for the second ship as it passes?

The temptation is to assume the speed add linearly. This is, after all, what happens at low speed. At speeds close to the speed of light, though, this could produce relative motion with speeds that appear to be greater than the speed of light, so something has to give.

As obscure as this is going to seem, and giving no other justification than the fact that Einstein's physics is not predicated on Euclidian geometry but, rather, Minkowskian geometry where $a^2 + b^2 \neq c^2$ but, rather, $a^2 - b^2 = c^2$, the relationship between the relative velocities of ships passing one another is:

$$w = \frac{u - v}{\left(1 - \frac{uv}{c^2}\right)}$$

So let's say that space ship A moving at .9c relative to "fixed space" passes a ship B moving in the opposite direction with velocity .9c relative to "fixed space." What speed does the first ship register for the second ship as it passes?

Using our relationship, the relative speed will appear to be:

$$\begin{aligned} v_{rel} &= \frac{(.9c)-(-.9c)}{\left(1-\frac{(.9c)(-.9c)}{c^2}\right)} \\ &= \frac{1.8c}{1.81} \\ &= .9945c \end{aligned}$$

Fun, eh?

Looking more closely at the scenario: You are in ship A and you see ship B pass you at .9945c. How much time do you observe passing in ship B during 10 of your seconds, and how long does ship B appear to be as it passes you if you know that their meter stick would measure their ship to be 100 meters long?